

## UNIT – V: Solving Two-Person and Zero-Sum Game

Game theory deals with decision situations in which two *intelligent* opponents with conflicting objectives are trying to outdo one another. Typical examples include launching advertising campaigns for competing products and planning strategies for warring armies.

In a game conflict, two opponents, known as players, will each have a (finite or infinite) number of alternatives or strategies. Associated with each pair of strategies is a payoff that one player receives from the other. Such games are known as two-person zero-sum games because a gain by one player signifies an equal loss to the other. It suffices, then, to summarize the game in terms of the payoff to one player. Designating the two players as  $A$  and  $B$  with  $m$  and  $n$  strategies, respectively, the game is usually represented by the payoff matrix to player  $A$  as

	$B_1$	$B_2$	-----	$B_n$
$A_1$	$a_{11}$	$a_{12}$	-----	$a_{1n}$
$A_2$	$a_{21}$	$a_{22}$	-----	$a_{2n}$
-----	-----	-----	-----	-----
$A_m$	$A_{m1}$	$a_{12}$	-----	$a_{mn}$

The representation indicates that if  $A$  uses strategy  $i$  and  $B$  uses strategy  $j$ , the payoff to  $A$  is  $a_{ij}$  which means that the payoff to  $B$  is  $-a_{ij}$

Two-person zero-sum games may be deterministic or probabilistic. The deterministic games will have saddle points and pure strategies exist in such games. In contrast, the probabilistic games will have no saddle points and mixed strategies are taken with the help of probabilities.

### **Definition of saddle point**

A saddle point of a matrix is the position of such an element in the payoff matrix, which is minimum in its row and the maximum in its column.

### **Procedure to find the saddle point**

- Select the minimum element of each row of the payoff matrix and mark them with circles.
- Select the maximum element of each column of the payoff matrix and mark them with squares.
- If their appears an element in the payoff matrix with a circle and a square together then that position is called saddle point and the element is the value of the game.

### **Solution of games with saddle point**

To obtain a solution of a game with a saddle point, it is feasible to find out

- Best strategy for player  $A$

- Best strategy for player B
- The value of the game

The best strategies for player A and B will be those which correspond to the row and column respectively through the saddle point.

### Optimal Solution of Two-Person Zero-Sum Games

Because games are rooted in conflict of interest, the optimal solution selects one or more strategies for each player such that any change in the chosen strategies does not improve the payoff to either player. These solutions can be in the form of a single pure strategy or several strategies mixed according to specific probabilities. The following two examples demonstrate the two cases:-

Two companies, A and B, sell two brands of flu medicine. Company A advertises in radio (A1), television (A2), and newspapers (A3). Company B, in addition to using radio (B1), television (B2), and newspapers (B3), also mails brochures (B4). Depending on the effectiveness of each advertising campaign, one company can capture a portion of the market from the other. The following matrix summarizes the percentage of the market captured or lost by company A.

		<i>Player B</i>				
		<i>B1</i>	<i>B2</i>	<i>B3</i>	<i>B4</i>	
<b>Player A</b>		<b>Radio</b>	<b>TV</b>	<b>Newspaper</b>	<b>Social media</b>	
	<b>A1</b>	<b>Radio</b>	<b>8</b>	<b>-2</b>	<b>9</b>	<b>-3</b>
	<b>A2</b>	<b>TV</b>	<b>6</b>	<b>5</b>	<b>6</b>	<b>8</b>
	<b>A3</b>	<b>Newspaper</b>	<b>-2</b>	<b>4</b>	<b>-9</b>	<b>5</b>

The solution of the game is based on the principle of securing the *best of the worst* for each player. If Company A selects strategy A1, then regardless of what B does, the worst that can happen is that A loses 3% of the market share to B. This is represented by the minimum value of the entries in row 1. Similarly, the strategy A2 worst outcome is for A to capture 5% of the market from B, and the strategy A3 worst outcome is for A to lose 9% to B. These results are listed in the "row min" column. To achieve the *best of the worst*, Company A chooses strategy A2 because it corresponds to the maximin value, or the largest element in the "row min" column. Next, consider Company B's strategy. Because the given payoff matrix is for A, B's *best of the worst* criterion requires determining the minimax value. The result is that Company B should select strategy B2.

The optimal solution of the game calls for selecting strategies A2 and B2, which means that both companies should use television advertising. The payoff will be in favor of company A, because its market share will increase by 5%. In this case, we say that the value of the game is 5%, and that A and B are using a saddle-point solution.

The saddle-point solution precludes the selection of a better strategy by either company. If B moves to another strategy ( $B1 > B3$ , or  $B4$ ), Company A can stay with strategy A2, which ensures that B will lose a worse share of the market (6% or 8%). By the same token, A does not want to use a different strategy because if A moves to strategy A3, B can move to B3 and realize a 9% increase in market share. A similar conclusion is realized if A moves to A1, as B can move to B4 and realize a 3% increase in market share.

The optimal saddle-point solution of a game need not be a pure strategy. Instead, the solution may require mixing two or more strategies randomly, as the following example illustrates.

Two players,  $A$  and  $B$ , play the coin-tossing game. Each player, unbeknownst to the other, chooses a head ( $H$ ) or a tail ( $T$ ). Both players would reveal their choices simultaneously. If they match ( $HH$  or  $TT$ ), player  $A$  receives \$1 from  $B$ . Otherwise,  $A$  pays  $B$  \$1.

The following payoff matrix for player  $A$  gives the row-min and the column-max values corresponding to  $A$ 's and  $B$ 's strategies, respectively.

		$B_H$	$B_T$	Row min
$A_H$	1	-1		-1
$A_T$	-1	1		-1
Col max	1	1		

The maximin and the minimax values of the games are - \$1 and \$1, respectively. Because the two values are not equal, the game does not have a pure strategy solution. In particular, if  $A_H$  is used by player  $A$ , player  $B$  will select  $B_T$  to receive \$1 from  $A$ . If this happens,  $A$  can move to strategy  $A_T$  to reverse the outcome of the game by receiving \$1 from  $B$ . The constant temptation to switch to another strategy shows that a pure strategy solution is not acceptable.

Instead, both players can randomly mix their respective pure strategies. In this case, the optimal value of the game will occur somewhere between the maximin and the minimax values of the game—that is,

$$\text{maximin (lower) value} \leq \text{value of the game} \leq \text{minimax (upper) value}$$

Thus, in the coin-tossing example, the value of the game must lie between -\$1 and +\$1.

## Examples

### Solve the payoff matrix

1.

		Player B		
		$B_1$	$B_2$	$B_3$
Player A	$A_1$	2	4	5
	$A_2$	10	7	9
	$A_3$	4	5	6

### Solution

		Player B			
		B 1	B 2	B 3	
Player A	A1	(2)	4	5	2
	A2	10	(7)	9	(7) Maximin value
	A3	(1)	5	6	4
		10	(7) Minimax value	9	

Strategy of player A – A2

Strategy of player B – B2

Value of the game = 7

2.

		Player B				
		I	II	III	IV	V
Player A	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

### Solution

		Player B					
		I	II	III	IV	V	
Player A	I	(-2)	0	0	5	3	-2
	II	3	2	(1)	2	2	(1) Maximin value
	III	(-4)	-3	0	-2	6	-4
	IV	5	3	-4	2	(-6)	-6
		5	3	(1) Minimax value	5	6	

Strategy of player A – II

Strategy of player B - III

Value of the game = 1

3..

	B1	B2	B3	B4
A1	1	7	3	4
A2	5	6	4	5
A3	7	2	0	3

**Solution**

	B1	B2	B3	B4	
A1	①	7	3	4	1
A2	5	6	④	5	④ Maximin value
A3	7	2	0	3	0
	7	7	④ Minimax value	5	

Strategy of player A – A2

Strategy of player B – B3

Value of the game = 4

4.

		B's Strategy				
		B1	B2	B3	B4	B5
A's Strategy	A1	8	10	-3	-8	-12
	A2	3	6	0	6	12
	A3	7	5	-2	-8	17
	A4	-11	12	-10	10	20
	A5	-7	0	0	6	2

**Solution**

		B's Strategy					
		B1	B2	B3	B4	B5	
A's strategy	A1	8	10	-5	-8	-12	12
	A2	3	6	0	6	-2	0 Maximin value
	A3	7	5	3	8	-7	-8
	A4	-1	-2	-10	10	20	11
	A5	7	0	5	6	2	-7
		8	12	0 Minimax value	10	20	

Strategy of player A – A2

Strategy of player B – B3

Value of the game = 0

5.

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

**Solution**

9	3	1	8	0	0
6	5	4	6	7	4 Maximin value
2	4	3	3	8	2
5	6	2	2	1	1
9	6	4	8	8	
Minimax value					

Value of the game = 4

## Exercise

1. Explain the concept of game theory.
2. What is a rectangular game?
3. What is a saddle point?
4. Define pure and mixed strategy in a game.
5. What are the characteristics of game theory?
6. Explain two-person zero-sum game giving suitable examples.
7. What are the limitations of game theory?
8. Explain the following terms
  - a. Competitive Game
  - b. Strategy
  - c. Value of the game
  - d. Pay-off-matrix
  - e. Optimal strategy
9. Explain Maximin and Minimax used in game theory
10. For the game with payoff matrix

$$\begin{array}{c} \text{Player B} \\ \text{Player A} \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

Determine the best strategies for player A and B and also the value of the game.